LOCAL ANTIMAGIC VERTEX COLORING OF A GRAPH

A SYNOPSIS

Submitted by

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(Reg. No. 201513124)

in partial fulfillment for award of the degree

of

DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

SCHOOL OF ADVANCED SCIENCES

KALASALINGAM ACADEMY OF RESEARCH AND EDUCATION

(DEEMED TO BE UNIVERSITY)

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TAMILNADU, INDIA

SEPTEMBER 2023

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This thesis embodies the work done by K. Premalatha under the guidance of Professor S. Arumugam.

The thesis consists of six chapters.

1. Preliminaries
2. Local antimagic vertex coloring of a graph
3. Local antimagic chromatic number of trees
4. Local antimagic chromatic number of cycle related join graphs
5. Local antimagic chromatic number of the join of paths and cycles with K_n or $\overline{K_n}$
6. Conclusion

By a graph $G = (V, E)$, we mean a finite undirected graph with neither loops nor multiple edges. The order $|V|$ and size $|E|$ of G are denoted by p and q respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1].

In Chapter 1 we collect some basic definitions and theorems in graphs and graph labelings which are needed for the subsequent chapters. Labelings and colorings are two major research

areas within graph theory. For a dynamic survey of graph labeling with an extensive bibliography, one may refer to Gallian [2]. Some of the well studied graph labelings are graceful labeling, harmonious labeling, equitable labeling, cordial labeling, arithmetic labeling, set labeling, radio labeling, prime labeling, magic labeling and antimagic labeling. For applications of labelings we refer to [3, 4, 5] and [6].

A labeling of a graph G is a mapping that assigns integers to the vertices or edges or both, subject to certain conditions. The labeling is called a vertex labeling or an edge labeling or a total labeling according to whether the domain of the mapping is $V(G)$ or $E(G)$ or $V(G) \cup E(G)$. Using such a labeling we compute the weight of a vertex or an edge through a specific rule. If all the computed weights are equal, then the labeling is *magic*. If all the computed weights are distinct, then the labeling is *antimagic*. If all the computed weights form an arithmetic progression with first term a and common difference d, then the labeling is called an (a, d)-*antimagic labeling*. Several authors have studied the properties of various types of magic and antimagic labelings.

For a detailed study of various types of magic labeling, we refer to the book by Wallis [7].

The notion of antimagic labeling was introduced by Hartsfield and Ringel [8].

Let $G = (V, E)$ be a (p, q)-graph. Let $f : E \to \{1, 2, \ldots, q\}$ be a bijection. The weight $f^+(v)$ of a vertex v is the sum of the labels of the edges incident with v. The labeling f is called an antimagic labeling if the weights of all the vertices are distinct. A graph G which admits an antimagic labeling is called *antimagic*.

Hartsfield and Ringel proposed the following conjecture.

Conjecture 1. *Every connected graph G different from K_2 is antimagic.*

Even if we restrict ourselves to trees, it is not known whether the conjecture is true.

A variety of antimagic labelings with lot of open problems are given in Bača and Miller [9].

In this thesis we introduce a new concept called local antimagic labeling which is a relaxed version of the concept of antimagic labeling. Any local antimagic labeling induces a vertex coloring of G and leads to a new coloring parameter, namely, the local antimagic chromatic number of G. This coloring parameter is denoted by $\chi_{la}(G)$. The focus of this thesis is a detailed investigation of this parameter.

Let $G = (V, E)$ be a connected graph of order p and size q and $G \neq K_2$. A bijection $f : E \to \{1, 2, \ldots, q\}$ is called a local antimagic labeling if for all $uv \in E$ we have $w(u) \neq w(v)$ where

$w(u) = \sum_{e \in E(u)} f(e)$. A graph G is local antimagic if G has a local antimagic labeling.

Any local antimagic labeling induces a proper vertex coloring of G where the vertex v is assigned the color $w(v)$. This naturally leads to the following concept.

The local antimagic chromatic number $\chi_{la}(G)$ is defined to be the minimum number of colors taken over all colorings of G induced by local antimagic labelings of G.

In Chapter 2 we present several basic results on local antimagic chromatic number of a graph.

Theorem 1. *For any tree T with l leaves, $\chi_{la}(T) \geq l + 1$.*

Theorem 2. *For the path $P_n = (v_1, v_2, \ldots, v_n)$ with $n \geq 3$, we have $\chi_{la}(P_n) = 3$.*

Theorem 3. *For the cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$ we have $\chi_{la}(C_n) = 3$.*

Theorem 4. *For the friendship graph F_n, we have $\chi_{la}(F_n) = 3$.*

Theorem 5. *For the wheel W_n of order $n + 1$ we have*

$$\chi_{la}(W_n) = \begin{cases} 4 & \text{if } n \equiv 1, 3 \ (mod\ 4) \\ 3 & \text{if } n \equiv 2 \ (mod\ 4). \end{cases}$$

The basic results presented in this chapter lead to the following problems which arise naturally.

Problem 1. *Characterize the class of graphs G for which* $\chi_{la}(G) = |V(G)| = p$.

Problem 2. *Characterize the class of graphs G for which* $\chi_{la}(G) = \chi(G)$.

Problem 3. *Does there exist a graph G of order p such that* $\chi_{la}(G) = p - k$ *for every* $k = 0, 1, 2, \ldots, p-2$?

In Chapter 3 we concentrate on the local antimagic chromatic number of the corona $G_1 \circ G_2$. We determine the value of χ_{la} for several families of trees.

Let $G = P_n \circ K_1$. If $n = 2$, then $G = P_4$ and $\chi_{la}(G) = 3$. Also $\chi_{la}(P_3 \circ K_1) = 4$.

Theorem 6. *Let $G = P_n \circ K_1$ where $n \geq 4$. Then $\chi_{la}(G) = n + 2$.*

Through a sequence of three lemmas we get the following theorem.

Theorem 7. *Let $G = P_n \circ \overline{K_m}$ where $n \geq 2$ and $m \geq 2$. Then $\chi_{la}(G) = mn + 2$.*

For a tree T with l leaves, $\chi_{la}(T) \geq l + 1$. For most of the trees considered in this chapter, $\chi_{la}(T) = l + 2$. Hence the following problems arise naturally.

Problem 4. *Characterize trees with* $\chi_{la}(T) = l + 1$

Problem 5. *Characterize trees with* $\chi_{la}(T) = l + 2$.

In Chapter 4 we investigate the local antimagic chromatic number of the join of two graphs.

The join of two graphs G and H is the graph obtained by joining each vertex of G to each vertex of H and it is denoted by $G \vee H$. Thus $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$. The graph $C_n \vee K_1$ is the wheel on $n + 1$ vertices and is denoted by W_n. The graph $P_n \vee K_1$ is the fan graph on $n + 1$ vertices and is denoted by F_n. In this chapter, we present a sharp upper bound for the local antimagic chromatic number $\chi_{la}(G \vee \overline{K_n})$ and sufficient conditions for equality. Using this we determine the value of $\chi_{la}(G \vee H)$ where $H = \overline{K_n}$ or C_n and G is a wheel related graph with $|V(G)| \equiv |V(H)|(mod\ 2)$.

We present several fundamental results on $\chi_{la}(G \vee H)$ where $H = \overline{K_r}$ or the cycle C_r. Similar investigation can be taken up for other choices of H such as path P_r, Star $K_{1,r-1}$ and complete graph K_r.

In Chapter 5 we continue with the study of local antimagic chromatic number of the join of two graphs. The following are the significant results presented in this chapter.

Theorem 8. *For* $m, n \geq 1$, $\chi_{la}(P_{2m} \vee \overline{K_{2n}}) = 3$.

Theorem 9. *For* $m, n \geq 1$, $\chi_{la}(P_{2m} \vee \overline{K_{2n-1}}) = 3$ *except that* $\chi_{la}(P_4 \vee \overline{K_1}) = 4$.

Theorem 10. *For $m \geq 1$, $n \geq 2$, $\chi_{la}(P_{2m} \vee C_{2n-1}) = 5$.*

Theorem 11. *For $m \geq 1$, $n \geq 1$, $\chi_{la}(P_{2m} \vee K_{2n}) = 2n + 2$.*

Theorem 12. *For $m \geq 1$, $n \geq 1$, $\chi_{la}(P_{2m} \vee K_{2n-1}) = 2n + 1$ except that $\chi_{la}(P_4 \vee K_1) = 4$.*

Theorem 13. *For $m \geq 2, n \geq 1$, $\chi_{la}(C_{2m} \vee \overline{K_{2n-1}}) = 3$.*

Theorem 14. *For $m, n \geq 2$, $\chi_{la}(C_{2m} \vee C_{2n-1}) = 5$.*

Theorem 15. *For $m \geq 2, n \geq 1$, $\chi_{la}(C_{2m} \vee K_{2n-1}) = 2n + 1$.*

Theorem 16. *For $m \geq 2, n \geq 1$, $\chi_{la}(K_{2n} \vee C_{2m-1}) = 2n + 3$.*

The results presented in this chapter naturally lead to the following problems for further investigation.

Problem 6. *For $m, n \geq 2$, determine $\chi_{la}(P_{2m-1} \vee \overline{K_n})$.*

Problem 7. *For $m \geq 1, n \geq 2$, determine $\chi_{la}(P_{2m} \vee C_{2n})$.*

Problem 8. *For $m, n \geq 2$, determine $\chi_{la}(C_{2m} \vee C_{2n})$.*

Problem 9. *For $m, n \geq 2$, determine $\chi_{la}(C_{2m-1} \vee \overline{K_{2n}})$.*

Various types of antimagic labelings have been extensively investigated and we refer to the book by Bača and Miller [9] for an excellent survey of such labelings. For each type of such antimagic labeling, the local version and the corresponding chromatic number can be studied. A brief summary of the progress and contributions made by several authors in this direction are presented in Chapter 6, which is the concluding chapter of this thesis.

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